

HIGHER ORDER SYMMETRIES OF REAL HYPERSURFACES IN \mathbb{C}^3

MARTIN KOLAR AND FRANCINE MEYLAN

(Communicated by Franc Forstneric)

ABSTRACT. We study nonlinear automorphisms of Levi degenerate hypersurfaces of finite multitype. By results of Kolar, Meylan, and Zaitsev in 2014, the Lie algebra of infinitesimal CR automorphisms may contain a graded component consisting of nonlinear vector fields of arbitrarily high degree, which has no analog in the classical Levi nondegenerate case, or in the case of finite type hypersurfaces in \mathbb{C}^2 . We analyze this phenomenon for hypersurfaces of finite Catlin multitype with holomorphically nondegenerate models in complex dimension three. The results provide a complete classification of such manifolds. As a consequence, we show on which hypersurfaces 2-jets are not sufficient to determine an automorphism. The results also confirm a conjecture about the origin of nonlinear automorphisms of Levi degenerate hypersurfaces, formulated by the first author for an AIM workshop in 2010.

1. INTRODUCTION

One of the central problems in CR geometry is the classification of real hypersurfaces in \mathbb{C}^n , up to biholomorphic equivalence. A complete solution of this problem should also lead to a complete understanding of automorphism groups of such manifolds.

When the hypersurface is Levi nondegenerate, the problem is well understood thanks to the classical work of Chern and Moser [11]. In particular, the infinitesimal CR automorphisms of such manifolds form a graded Lie algebra with at most five components. Moreover, by results of Kruzhilin and Loboda ([29]), if a strongly pseudoconvex hypersurface is not equivalent to the sphere, there are at most three graded components, and all infinitesimal automorphisms are linear in appropriate coordinates. For the sphere itself, the coefficients are at most quadratic, which implies 2-jet determination in general.

Similar results were obtained for hypersurfaces of finite type in \mathbb{C}^2 . In particular, the same 2-jet determination result holds (see [16], [24]).

In a recent paper [23], the same problem is considered for Levi degenerate hypersurfaces in \mathbb{C}^n with weighted homogeneous polynomial models, which replace

Received by the editors August 10, 2015 and, in revised form, January 11, 2016.

2010 *Mathematics Subject Classification*. Primary 32V35, 32V40.

The first author was supported by the project CZ.1.07/2.3.00/20.0003 of the Operational Programme Education for Competitiveness of the Ministry of Education, Youth and Sports of the Czech Republic.

The second author was supported by Swiss NSF Grant 2100-063464.00/1.

the model hyperquadric from the nondegenerate case. The results describe possible structures of infinitesimal CR automorphism algebras for hypersurfaces of finite Catlin multitype with holomorphically nondegenerate models. Compared to the Levi nondegenerate case, there are in general six possible components. The new phenomenon is the existence of nonlinear infinitesimal CR automorphisms in the complex tangential variables, which are of arbitrarily high degree in general.

Our aim in this paper is to analyze this phenomenon and provide a complete description of hypersurfaces of finite Catlin multitype with holomorphically nondegenerate models in \mathbb{C}^3 which admit such automorphisms.

Let us recall that the Catlin multitype is an important CR invariant which Catlin introduced to prove subelliptic estimates on pseudoconvex domains ([9], [10]). The definition of multitype was extended to the general case (not necessarily pseudoconvex) in [25]. It provides a natural setting for an extension of the Chern-Moser theory to degenerate manifolds ([23]).

We consider a weighted homogeneous model of finite Catlin multitype that is holomorphically nondegenerate. Let

$$(1.1) \quad M_P := \{\operatorname{Im} w = P(z, \bar{z})\}, \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C},$$

where P is a real-valued weighted homogeneous polynomial with respect to the multitype weights μ_1, μ_2 (see Section 2 for the needed definitions).

As proved in [23], the Lie algebra of infinitesimal CR automorphisms $\mathfrak{g} = \operatorname{aut}(M_P, 0)$ of M_P admits the weighted decomposition given by

$$(1.2) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^2 \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_1.$$

Here \mathfrak{g}_c contains vector fields commuting with W and \mathfrak{g}_n contains vector fields not commuting with W , both of weight $\mu \in (0, 1)$ (see [23] for more details). Notice that \mathfrak{g}_{-1} contains $W = \partial_w$ and \mathfrak{g}_0 contains the weighted Euler field, hence they are always nontrivial. A complete description of \mathfrak{g}_1 is also contained in [23].

Remark 1.1. By the results of [23], the elements of \mathfrak{g}_n and \mathfrak{g}_1 are determined by ordinary 2-jets, hence *higher order* infinitesimal automorphisms may occur only when \mathfrak{g}_c is nontrivial.

In this paper, we study all hypersurfaces whose model has nontrivial \mathfrak{g}_c . Our results confirm a conjecture about the origin of nonlinear automorphisms of Levi degenerate hypersurfaces formulated by the first author (see the 2010 AIM list of problems at <http://www.aimath.org/WWN/crmappings/crmappings.pdf>): M_P has a nonlinear symmetry if and only if there is a holomorphic mapping f of M_P into a hyperquadric in \mathbb{C}^K and a symmetry of the hyperquadric, which is f -related to the automorphism of M_P .

Note that mappings of CR manifolds into hyperquadrics have been studied intensively in recent years (see e.g. [1], [14]). Here we ask in addition about compatibility with a symmetry of the hyperquadric. Let us remark that analyzing \mathfrak{g}_n requires completely different techniques and is the subject of a forthcoming paper [28].

In order to formulate our first result, let us recall that two vector fields X_1 and X_2 are f -related (or compatible by f) if $f_*(X_1) = X_2$.

Theorem 1.2. *Let M_P be a holomorphically nondegenerate hypersurface given by (1.1). Assume that $\dim \mathfrak{g}_c > 0$ and let $Y \in \mathfrak{g}_c$ be a nonzero vector field. Then there exist an integer $K \geq 3$ and a holomorphic polynomial mapping $f : \mathbb{C}^3 \rightarrow \mathbb{C}^K$ which maps M_P into a hyperquadric $H \subseteq \mathbb{C}^K$, such that Y is f -related with an infinitesimal CR automorphism of H .*

The proof is based on an explicit complete description of models with nontrivial \mathfrak{g}_c , for which we need the following definition.

Definition 1.3. Let Y be a weighted homogeneous vector field. A pair of finite sequences of holomorphic weighted homogeneous polynomials $\{U^1, \dots, U^n\}$ and $\{V^1, \dots, V^n\}$ is called a symmetric pair of Y -chains if

$$(1.3) \quad Y(U^n) = 0, \quad Y(U^j) = c_j U^{j+1}, \quad j = 1, \dots, n-1,$$

$$(1.4) \quad Y(V^n) = 0, \quad Y(V^j) = d_j V^{j+1}, \quad j = 1, \dots, n-1,$$

where c_j, d_j are nonzero complex constants which satisfy

$$(1.5) \quad c_j = -\bar{d}_{n-j}.$$

If the two sequences are identical, we say that $\{U^1, \dots, U^n\}$ is a symmetric Y -chain.

Example 1.4. Let

$$Y = iz_2^l \frac{\partial}{\partial z_1}.$$

Then the pair $\{U^1, U^2\} = \{z_1, z_2^l\}$ is a symmetric Y -chain, since $Y(U^2) = 0$ and $Y(U^1) = iU^2$. Then for the hypersurface given by

$$\operatorname{Im} w = \operatorname{Re} U^1 \overline{U^2} = \operatorname{Re} z_1 \overline{z_2^l},$$

we have $Y \in \mathfrak{g}_c$.

The following result shows that in general the elements of \mathfrak{g}_c arise in an analogous way.

Theorem 1.5. *Let M_P be a holomorphically nondegenerate model given by (1.1) admitting a nontrivial $Y \in \mathfrak{g}_c$. Then P can be decomposed in the following way:*

$$(1.6) \quad P = \sum_{j=1}^M T_j,$$

where each T_j is given by

$$(1.7) \quad T_j = \operatorname{Re} \left(\sum_{k=1}^{N_j} U_j^k \overline{V_j^{N_j-k+1}} \right),$$

where $\{U_j^1, \dots, U_j^{N_j}\}$ and $\{V_j^1, \dots, V_j^{N_j}\}$ are a symmetric pair of Y -chains.

Conversely, if Y and P satisfy (1.3) – (1.7), then $Y \in \mathfrak{g}_c$.

Remark 1.6. It is immediate to see that Y is uniquely and explicitly determined by P . More precisely, since M_P is holomorphically nondegenerate, at least one of the T_j has length $N_j \geq 2$. For such a T_j we have

$$Y = \frac{c_{N_j-1} U_j^{N_j-1}}{\Delta} \left(-\frac{\partial U_j^{N_j}}{\partial z_2} \partial_{z_1} + \frac{\partial U_j^{N_j}}{\partial z_1} \partial_{z_2} \right),$$

where Δ is the Jacobian of $\{U_j^{N_j-1}, U_j^{N_j}\}$. Hence for a given hypersurface the results also provide a simple constructive tool to determine \mathfrak{g}_c . Moreover, this also shows that the real dimension of \mathfrak{g}_c is at most one.

Examples of symmetric chains of arbitrary length are described at the end of Section 3. Using Remark 1.1, we obtain

Theorem 1.7. *Let M be an arbitrary smooth hypersurface of finite Catlin multiplicity. If its model is holomorphically nondegenerate and not biholomorphically equivalent to one of the form described in Theorem 1.5, then the automorphisms of M are determined by their 2-jets.*

The paper is organized as follows. Section 2 contains the necessary definitions used in the rest of the paper. Section 3 deals with the \mathfrak{g}_c component of the algebra $\text{aut}(M_P, 0)$. Section 4 completes the proofs of the above theorems.

2. PRELIMINARIES

Let $M \subseteq \mathbb{C}^3$ be a smooth hypersurface, and let $p \in M$ be a point of finite type $m \geq 2$ in the sense of Kohn and Bloom-Graham ([5], [6], [22]).

We consider local holomorphic coordinates (z, w) vanishing at p , where $z = (z_1, z_2)$ and $z_j = x_j + iy_j$, $j = 1, 2$, and $w = u + iv$. The hyperplane $\{v = 0\}$ is assumed to be tangent to M at p , hence M is described near $p = 0$ as the graph of a uniquely determined real-valued function

$$(2.1) \quad v = \varphi(z_1, z_2, \bar{z}_1, \bar{z}_2, u), \quad d\varphi(0) = 0.$$

We can assume that (see e.g. [5])

$$(2.2) \quad \varphi(z_1, z_2, \bar{z}_1, \bar{z}_2, u) = P_m(z, \bar{z}) + o(u, |z|^m),$$

where $P_m(z, \bar{z})$ is a nonzero homogeneous polynomial of degree m without pluriharmonic terms.

Recall that the definition of multitype involves rational weights associated to the variables w, z_1, z_2 . The variables w, u , and v are given weight one, reflecting our choice of tangential and normal variables. The complex tangential variables (z_1, z_2) are treated according to the following definitions (for more details, see [25]).

Definition 2.1. A weight is a pair of nonnegative rational numbers $\Lambda = (\lambda_1, \lambda_2)$, where $0 \leq \lambda_j \leq \frac{1}{2}$ and $\lambda_1 \geq \lambda_2$.

Let $\Lambda = (\lambda_1, \lambda_2)$ be a weight, and let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ be multiindices. The weighted degree κ of a monomial

$$q(z, \bar{z}, u) = c_{\alpha\beta l} z^\alpha \bar{z}^\beta u^l, \quad l \in \mathbb{N},$$

is defined as

$$\kappa := l + \sum_{i=1}^2 (\alpha_i + \beta_i) \lambda_i.$$

A polynomial $Q(z, \bar{z}, u)$ is weighted homogeneous of weighted degree κ if it is a sum of monomials of weighted degree κ .

For a weight Λ , the weighted length of a multiindex $\alpha = (\alpha_1, \alpha_2)$ is defined by

$$|\alpha|_\Lambda := \lambda_1 \alpha_1 + \lambda_2 \alpha_2.$$

Similarly, if $\alpha = (\alpha_1, \alpha_2)$ and $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)$ are two multiindices, the weighted length of the pair $(\alpha, \hat{\alpha})$ is

$$|(\alpha, \hat{\alpha})|_{\Lambda} := \lambda_1(\alpha_1 + \hat{\alpha}_1) + \lambda_2(\alpha_2 + \hat{\alpha}_2).$$

Definition 2.2. A weight Λ will be called distinguished for M if there exist local holomorphic coordinates (z, w) in which the defining equation of M takes the form

$$(2.3) \quad v = P(z, \bar{z}) + o_{\Lambda}(1),$$

where $P(z, \bar{z})$ is a nonzero Λ -homogeneous polynomial of weighted degree 1 without pluriharmonic terms, and $o_{\Lambda}(1)$ denotes a smooth function whose derivatives of weighted order less than or equal to one vanish.

The fact that distinguished weights do exist follows from (2.2). For these coordinates (z, w) , we have

$$\Lambda = \left(\frac{1}{m}, \frac{1}{m}\right).$$

In the following, we shall consider the standard lexicographic order on the set of pairs.

We recall the following definition (see [9]).

Definition 2.3. Let $\Lambda_M = (\mu_1, \mu_2)$ be the infimum of all possible distinguished weights Λ with respect to the lexicographic order. The multitype of M at p is defined to be the pair

$$(m_1, m_2),$$

where

$$m_j = \begin{cases} \frac{1}{\mu_j} & \text{if } \mu_j \neq 0, \\ \infty & \text{if } \mu_j = 0. \end{cases}$$

If none of the m_j is infinity, we say that M is of finite multitype at p .

Clearly, since the definition of multitype includes all distinguished weights, the infimum is a biholomorphic invariant.

Coordinates corresponding to the multitype weight Λ_M , in which the local description of M has the form (2.3), with P being Λ_M -homogeneous, are called multitype coordinates.

If M is of finite multitype at p , the infimum in (2.3) is attained, which implies that multitype coordinates do exist ([9], [25]).

Definition 2.4. Let M be given by (2.3). We define a model hypersurface M_P associated to M at $p = 0$ by

$$(2.4) \quad M_P = \{(z, w) \in \mathbb{C}^3 \mid v = P(z, \bar{z})\}.$$

Next let us recall the following definitions.

Definition 2.5. Let X be a holomorphic vector field of the form

$$(2.5) \quad X = \sum_{j=1}^2 f^j(z, w) \partial_{z_j} + g(z, w) \partial_w.$$

We say that X is rigid if f^1, f^2, g are all independent of the variable w .

Note that the rigid vector field W , of homogeneous weight -1 , given by

$$(2.6) \quad W = \partial_w$$

lies in $\text{aut}(M_P, 0)$. We will denote by E the weighted homogeneous vector field of weight 0 defined by

$$(2.7) \quad E = \sum_{j=1}^2 \mu_j z_j \partial_{z_j} + w \partial_w,$$

i.e. the weighted Euler field. Note that by definition of μ_j , E is a nonrigid vector field lying in $\text{aut}(M_P, 0)$.

We can divide homogeneous rigid vector fields into three types, and introduce the following terminology.

Definition 2.6. Let $X \in \text{aut}(M_P, 0)$ be a rigid weighted homogeneous vector field. X is called

- (1) a shift if the weighted degree of X is less than zero;
- (2) a rotation if the weighted degree of X is equal to zero;
- (3) a generalized rotation if the weighted degree of X is bigger than zero.

Notice that $X \in \text{aut}(M_P, 0)$ is a generalized rotation if and only if it has positive weighted degree and commutes with W . In other words, generalized rotations are precisely the elements of \mathfrak{g}_c .

3. GENERALIZED ROTATIONS

In this section we study nonlinear infinitesimal CR automorphisms of hypersurfaces of finite multitype and derive an explicit description of all models which admit a generalized rotation.

Lemma 3.1. *Let $Y = f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2}$ be a weighted homogeneous holomorphic vector field of weighted degree > 0 . Then the space of weighted homogeneous polynomials in z of a given weighted degree ν annihilated by X has complex dimension at most one.*

Proof. First we claim that Y cannot be a multiple of the Euler field. Indeed, if $Y = hE$ with h holomorphic and nonconstant, then $Y(P) = 0$ has no homogeneous solution, since $\text{Re } Y(P) = \text{Re } hP \neq 0$. Hence there exists a point q such that $Y(q)$ is not a multiple of the Euler field. This point lies on a uniquely determined complex curve $z_1^{m_1} = cz_2^{m_2}$, and Y is transverse to this curve in a neighborhood of q . By homogeneity, on this curve $P(z_1, z_2)$ is determined up to a multiplicative complex constant. Fixing this constant, by the uniqueness property for solutions of complex ODEs ([19]), the equation $Y(P) = 0$ determines P uniquely in a neighborhood of q . Since P is a polynomial, if it exists, it is determined uniquely. Hence the space of solutions of $Y(P) = 0$ is at most one dimensional. \square

Lemma 3.2. *Let V_n , $n \in \mathbb{N}$, be the space*

$$(3.1) \quad V_n = \{X | Y^n(X) = 0\},$$

where X is a holomorphic polynomial of a given constant weighted length and Y is a weighted homogeneous holomorphic vector field. Then

$$(3.2) \quad \dim V_n \leq n.$$

Moreover, when $d_n = \dim V_n > 0$, one can choose a basis for V_n of the form

$$(3.3) \quad \{F_s^n, s = 1, 2, \dots, d_n | Y^{d_n}(F_{d_n}^n) = 0, \\ Y^{d_n-1}(F_{d_n}^n) \neq 0, Y^{d_n-1}(F_s^n) = 0, s = 1, 2, \dots, d_n - 1\}.$$

Proof. We prove the lemma by induction. The case $n = 1$ is a direct application of the previous lemma. Suppose now that the lemma is true for n and prove it for $n + 1$. We have

$$(3.4) \quad V_{n+1} = \{X | Y^{n+1}(X) = 0\} = \{X | Y^n(Y(X)) = 0\}.$$

By induction, we obtain that

$$(3.5) \quad Y(X) \in \text{span}[F_s^n, | Y^{d_n}(F_{d_n}^n) = 0, Y^{d_n}(F_{d_n}^n) \neq 0, Y^{d_n-1}(F_s^n) = 0, \\ s = 1, \dots, d_n - 1],$$

which implies that

$$(3.6) \quad \dim V_{n+1} \leq n + 1.$$

After performing a linear combination of the solutions X of (3.5), we may satisfy (3.3). \square

Theorem 3.3. Let M_P be given by (1.1) admitting a generalized rotation Y . Then P can be decomposed in the following way:

$$(3.7) \quad P = \sum_{j=1}^M T_j,$$

where each T_j is given by

$$(3.8) \quad T_j = \text{Re} \left(\sum_{k=1}^{N_j} U_j^k \overline{V_j^{N_j-k+1}} \right),$$

where $\{U_j^1, \dots, U_j^{N_j}\}$ and $\{V_j^1, \dots, V_j^{N_j}\}$ are a symmetric pair of Y -chains.

Proof. Let

$$(3.9) \quad P = \sum_{k=1}^l P_k$$

be the bihomogeneous expansion of P , where $P_1 \neq 0, P_l \neq 0$. Each P_j is weighted homogeneous with respect to z of weighted degree c_j where $c_1 < c_2 < \dots < c_l$.

We may write

$$(3.10) \quad P_1 = \sum_{j=1}^r S_j^{c_1} \overline{S_j^{\hat{c}_1}},$$

with r minimal. Note that $c_1 + \hat{c}_1 = 1$. We claim that $r = 1$. Since Y is a generalized rotation, we must have

$$(3.11) \quad \overline{Y} \left(\sum_{j=1}^r S_j^{c_1} \overline{S_j^{\hat{c}_1}} \right) = \sum_{j=1}^r S_j^{c_1} \overline{Y(S_j^{\hat{c}_1})} = 0.$$

Since r is minimal, this implies that

$$(3.12) \quad Y(S_j^{\hat{c}_1}) = 0$$

for all j . Using Lemma 3.2, we conclude that

$$(3.13) \quad S_j^{\hat{c}_1} \in [S_1^{\hat{c}_1}]$$

for all j . We may then write P_1 as

$$(3.14) \quad P_1 = Q_1^{c_1} \overline{Q_1^{\hat{c}_1}}.$$

Hence, $r = 1$ and the claim is proved. We write now

$$(3.15) \quad P_k = \sum_{j=1}^{r_k} S_j^{c_k} \overline{S_j^{\hat{c}_k}},$$

with r_k minimal.

We claim that P_k can be rewritten as

$$(3.16) \quad P_k = Q_k^{c_k} \overline{Q_k^{\hat{c}_k}} + \tilde{P}_k$$

so that there is a $d_k \leq k$ such that

$$(3.17) \quad \overline{Y^{d_k}}(\overline{Q_k^{\hat{c}_k}}) = 0, \quad \overline{Y^{d_k-1}}(\overline{Q_k^{\hat{c}_k}}) \neq 0, \quad \overline{Y^{d_k-1}}(\tilde{P}_k) = 0.$$

We prove the claim by induction. The case $k = 1$ has just been proved.

Suppose by induction that (3.16) holds for k . Since Y is a generalized rotation, we have

$$(3.18) \quad Y(Q_k^{c_k}) \overline{Q_k^{\hat{c}_k}} + Y(\tilde{P}_k) + \sum_{j=1}^{r_{k+1}} S_{k+1}^{c_{k+1}} \overline{Y(S_{k+1}^{\hat{c}_{k+1}})} = 0.$$

Applying $\overline{Y^{d_k}}$ to (3.18), we get

$$(3.19) \quad \sum_{j=1}^{r_{k+1}} S_j^{c_{k+1}} \overline{Y^{d_k+1}(S_j^{\hat{c}_{k+1}})} = 0.$$

Since r_{k+1} is minimal,

$$(3.20) \quad \overline{Y^{d_k+1}}(\overline{S_j^{\hat{c}_{k+1}}}) = 0$$

for all j . Using Lemma 3.2, we obtain that $r_{k+1} \leq d_k + 1 \leq k + 1$. Using (3.3), we may then rewrite P_{k+1} in the form given by (3.16). The claim is then proved.

Let $N_1 \leq l$ be minimal such that

$$Y(Q_k^{c_k}) \neq 0, \quad k = 1, \dots, N_1 - 1, \quad Y(Q_{N_1}^{c_{N_1}}) = 0.$$

We consider the following set E_1 given by

$$(3.21) \quad E_1 = \{Q_k^{c_k} \overline{Q_k^{\hat{c}_k}}, \quad k = 1, \dots, N_1\}.$$

Note that this set is not empty since $Y(Q_l^{c_l}) = 0$.

We claim that the following hold for every element of E_1 .

- (1) $d_k = k$, $k = 1, \dots, N_1$,
- (2) $Y(Q_k^{c_k}) = a_k Q_{k+1}^{c_{k+1}}$,
- (3) $Y(Q_{k+1}^{\hat{c}_{k+1}}) = b_{k+1} Q_k^{\hat{c}_k} + R_k$, where $Y^{k-1}(R_k) = 0$.

We show that $d_k = k$ using induction as above. Indeed, suppose that this is true for $k < N_1 - 1$ and show that it is also true for $k + 1$. Using the fact that Y is a generalized rotation, we have as in (3.18),

$$(3.22) \quad Y(Q_k^{c_k})\overline{Q_k^{\hat{c}_k}} + Y(\tilde{P}_k) + (Q_{k+1}^{c_{k+1}})\overline{Y(Q_{k+1}^{\hat{c}_{k+1}})} + \overline{Y}(\tilde{P}_{k+1}) = 0.$$

Applying \overline{Y}^{k-1} to (3.22), we obtain

$$(3.23) \quad Y(Q_k^{c_k})\overline{Y}^{k-1}(\overline{Q_k^{\hat{c}_k}}) + (Q_{k+1}^{c_{k+1}})\overline{Y}^k(\overline{Q_{k+1}^{\hat{c}_{k+1}}}) = 0.$$

Hence, using (3.23), $d_{k+1} = k + 1$ by definition of E_1 , and hence

$$(3.24) \quad Y(Q_k^{c_k}) = a_k Q_{k+1}^{c_{k+1}},$$

$$(3.25) \quad Y^k(Q_{k+1}^{\hat{c}_{k+1}}) = b_{k+1} Y^{k-1} Q_k^{\hat{c}_k},$$

which implies

$$(3.26) \quad Y^{k-1}(Y(Q_{k+1}^{\hat{c}_{k+1}}) - b_{k+1} Q_k^{\hat{c}_k}) = 0,$$

and hence

$$(3.27) \quad Y(Q_{k+1}^{\hat{c}_{k+1}}) = b_{k+1} Q_k^{\hat{c}_k} + R_k,$$

where $Y^{k-1}(R_k) = 0$. This achieves the proof of the claim. Using (3.27) and (3.16), we may then assume without loss of generality that $R_k = 0$. We define the chains by putting

$$(3.28) \quad \begin{cases} U_1^k := Q_k^{c_k}, \\ V_1^k := Q_k^{\hat{c}_{N_1-k+1}}. \end{cases}$$

It follows from the above properties of E_1 that U_1^k and V_1^k form a chain. In other words, we may write

$$(3.29) \quad P = \operatorname{Re} \left(\sum_{k=1}^{N_1} U_1^k \overline{V_1^{N_1-k+1}} \right) + \hat{P}, \quad k = 1, \dots, N_1.$$

It follows from (3.23) that Y is a generalized rotation for

$$\operatorname{Im} w = \operatorname{Re} \left(\sum_{k=1}^{N_1} U_1^k \overline{V_1^{N_1-k+1}} \right).$$

It follows from (3.23) that $a_k = -\bar{b}_{k+1}$, which means that the U and V are a pair of symmetric chains. Hence Y is a generalized rotation also for \hat{P} . We can repeat the above argument for \hat{P} and in a finite number of steps we reach the conclusion of the theorem. \square

Note that symmetric chains and pairs of chains of any length can arise.

Example 3.4. Let

$$Y = z_1^2 \partial_{z_1} - z_1 z_2 \partial_{z_2}.$$

Given three integers $1 \leq l \leq m \leq n$, we first define

$$U^l = z_1^n z_2^n.$$

We can build a symmetric Y -chain by setting $U^j = c_j z_1^{n-l+j} z_2^n$ for $j = 1, \dots, l-1$ for suitable constants c_j . Analogously, setting in addition

$$V^l = z_1^m z_2^m,$$

we can get in the same way a pair of symmetric Y -chains of arbitrary length l .

4. PROOFS OF THE MAIN RESULTS

In this section we complete the proofs of the results stated in the Introduction. The first part of Theorem 1.5 has already been proved in Section 3 (as Theorem 3.3). The second, converse part of the statement is immediate to verify.

In order to prove Theorem 1.7, we combine Theorem 1.5 with Theorem 4.7 and Theorem 6.2 from [23]. They imply that on a smooth hypersurface of finite Catlin multitype, 2-jets are always sufficient to determine an element from \mathfrak{g}_1 and \mathfrak{g}_n .

Proof of Theorem 1.2. In the notation of Theorem 3.3, we set

$$(4.1) \quad K = 2 \sum_{j=1}^M N_j + 1.$$

We define a hyperquadric in \mathbb{C}^{K+1} by

$$(4.2) \quad \operatorname{Im} \eta = \operatorname{Re} \sum_{j=1}^M \sum_{k=1}^{N_j} \zeta_{j,k} \overline{\zeta'_{j,N_j-k+1}},$$

and consider the mapping $\mathbb{C}^3 \rightarrow \mathbb{C}^{K+1}$ given by $\eta = w$ and

$$(4.3) \quad \zeta_{j,k} = U_j^k(z_1, z_2)$$

and

$$(4.4) \quad \zeta'_{j,k} = V_j^k(z_1, z_2).$$

It is immediate to verify that the automorphism Y of M_P is f -related to the automorphism of this hyperquadric, defined by

$$(4.5) \quad Z = \sum_{j=1}^M \sum_{k=2}^{N_j} c_{k-1,j} \zeta_{j,k} \partial_{\zeta_{j,k-1}} + d_{k-1,j} \zeta'_{j,k} \partial_{\zeta'_{j,k-1}}.$$

Indeed, the condition for f -related vector fields becomes exactly the chain condition (1.3)-(1.5).

REFERENCES

- [1] M. S. Baouendi, Peter Ebenfelt, and Xiaojun Huang, *Super-rigidity for CR embeddings of real hypersurfaces into hyperquadrics*, Adv. Math. **219** (2008), no. 5, 1427–1445, DOI 10.1016/j.aim.2008.07.001. MR2458142 (2010b:32051)
- [2] E. Bedford and S. I. Pinchuk, *Convex domains with noncompact groups of automorphisms* (Russian, with Russian summary), Mat. Sb. **185** (1994), no. 5, 3–26, DOI 10.1070/SM1995v082n01ABEH003550; English transl., Russian Acad. Sci. Sb. Math. **82** (1995), no. 1, 1–20. MR1275970 (95e:32037)
- [3] V. K. Beloshapka, V. V. Ezhov, and G. Schmalz, *Holomorphic classification of four-dimensional surfaces in \mathbb{C}^3* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **72** (2008), no. 3, 3–18, DOI 10.1070/IM2008v072n03ABEH002406; English transl., Izv. Math. **72** (2008), no. 3, 413–427. MR2432751 (2009g:32074)

- [4] V. K. Beloshapka and I. G. Kossovskiy, *Classification of homogeneous CR-manifolds in dimension 4*, J. Math. Anal. Appl. **374** (2011), no. 2, 655–672, DOI 10.1016/j.jmaa.2010.07.008. MR2729251 (2011j:32053)
- [5] Thomas Bloom and Ian Graham, *On “type” conditions for generic real submanifolds of \mathbf{C}^n* , Invent. Math. **40** (1977), no. 3, 217–243. MR0589930 (58 #28644)
- [6] Thomas Bloom and Ian Graham, *A geometric characterization of points of type m on real submanifolds of \mathbf{C}^n* , J. Differential Geometry **12** (1977), no. 2, 171–182. MR0492369 (58 #11495)
- [7] Michael Beals, Charles Fefferman, and Robert Grossman, *Strictly pseudoconvex domains in \mathbf{C}^n* , Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 2, 125–322, DOI 10.1090/S0273-0979-1983-15087-5. MR684898 (85a:32025)
- [8] Cartan, E., *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I*, Ann. Math. Pura Appl. **11** (1932), 17–90.
- [9] David Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) **120** (1984), no. 3, 529–586, DOI 10.2307/1971087. MR769163 (86c:32019)
- [10] David Catlin, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), no. 1, 131–191, DOI 10.2307/1971347. MR898054 (88i:32025)
- [11] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271. MR0425155 (54 #13112)
- [12] John P. D’Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **115** (1982), no. 3, 615–637, DOI 10.2307/2007015. MR657241 (84a:32027)
- [13] Peter Ebenfelt, *New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy*, J. Differential Geom. **50** (1998), no. 2, 207–247. MR1684982 (2000a:32077)
- [14] Peter Ebenfelt, Xiaojun Huang, and Dmitri Zaitsev, *The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics*, Amer. J. Math. **127** (2005), no. 1, 169–191. MR2115664 (2006c:32047)
- [15] Peter Ebenfelt, Bernhard Lamel, and Dmitri Zaitsev, *Degenerate real hypersurfaces in \mathbf{C}^2 with few automorphisms*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 3241–3267, DOI 10.1090/S0002-9947-09-04626-1. MR2485425 (2010f:32037)
- [16] P. Ebenfelt, B. Lamel, and D. Zaitsev, *Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case*, Geom. Funct. Anal. **13** (2003), no. 3, 546–573, DOI 10.1007/s00039-003-0422-y. MR1995799 (2004g:32033)
- [17] Gregor Fels and Wilhelm Kaup, *Classification of Levi degenerate homogeneous CR-manifolds in dimension 5*, Acta Math. **201** (2008), no. 1, 1–82, DOI 10.1007/s11511-008-0029-0. MR2448066 (2009k:32045)
- [18] Xiaojun Huang and Wanke Yin, *A Bishop surface with a vanishing Bishop invariant*, Invent. Math. **176** (2009), no. 3, 461–520, DOI 10.1007/s00222-008-0167-1. MR2501295 (2010f:32039)
- [19] Yulij Ilyashenko and Sergei Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008. MR2363178 (2009b:34001)
- [20] A. V. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. **146** (1999), no. 1, 1–38, DOI 10.1006/aima.1998.1821. MR1706680 (2000i:32053)
- [21] Howard Jacobowitz, *An introduction to CR structures*, Mathematical Surveys and Monographs, vol. 32, American Mathematical Society, Providence, RI, 1990. MR1067341 (93h:32023)
- [22] J. J. Kohn, *Boundary behavior of δ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geometry **6** (1972), 523–542. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays. MR0322365 (48 #727)
- [23] Martin Kolar, Francine Meylan, and Dmitri Zaitsev, *Chern-Moser operators and polynomial models in CR geometry*, Adv. Math. **263** (2014), 321–356, DOI 10.1016/j.aim.2014.06.017. MR3239141
- [24] Martin Kolář and Francine Meylan, *Infinitesimal CR automorphisms of hypersurfaces of finite type in \mathbf{C}^2* , Arch. Math. (Brno) **47** (2011), no. 5, 367–375. MR2876940 (2012m:32038)
- [25] Martin Kolář, *The Catlin multitype and biholomorphic equivalence of models*, Int. Math. Res. Not. IMRN **18** (2010), 3530–3548, DOI 10.1093/imrn/rnq013. MR2725504 (2011m:32066)

- [26] Martin Kolář, *Normal forms for hypersurfaces of finite type in \mathbb{C}^2* , Math. Res. Lett. **12** (2005), no. 5-6, 897–910, DOI 10.4310/MRL.2005.v12.n6.a10. MR2189248 (2007d:32034)
- [27] Martin Kolář and Francine Meylan, *Chern-Moser operators and weighted jet determination problems*, Geometric analysis of several complex variables and related topics, Contemp. Math., vol. 550, Amer. Math. Soc., Providence, RI, 2011, pp. 75–88, DOI 10.1090/conm/550/10867. MR2868555
- [28] M. Kolář and F. Meylan, *Automorphism groups of Levi degenerate hypersurfaces in \mathbb{C}^3* , preprint
- [29] N. G. Kruzhilin and A. V. Loboda, *Linearization of local automorphisms of pseudoconvex surfaces* (Russian), Dokl. Akad. Nauk SSSR **271** (1983), no. 2, 280–282. MR718188 (85c:32052)
- [30] Poincaré, H., *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo **23** (1907), 185–220.
- [31] Nancy K. Stanton, *Infinitesimal CR automorphisms of real hypersurfaces*, Amer. J. Math. **118** (1996), no. 1, 209–233. MR1375306 (97h:32027)
- [32] A. G. Vitushkin, *Real-analytic hypersurfaces of complex manifolds* (Russian), Uspekhi Mat. Nauk **40** (1985), no. 2(242), 1–35, 237. MR786085 (86h:32031)
- [33] S. M. Webster, *On the Moser normal form at a non-umbilic point*, Math. Ann. **233** (1978), no. 2, 97–102, DOI 10.1007/BF01421918. MR486511 (80b:32008)

DEPARTMENT OF MATHEMATICS AND STATISTICS, MASARYK UNIVERSITY, KOTLARSKA 2, 611 37 BRNO, CZECH REPUBLIC

E-mail address: `mkolar@math.muni.cz`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FRIBOURG, CH 1700 PEROLLES, FRIBOURG, SWITZERLAND

E-mail address: `francine.meylan@unifr.ch`